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NON-CONVEX PERTURBATIONS OF MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS

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ABSTRACT

We give an existence result for

$$\dot{x} \in -Ax + F(x)$$

where A is a maximal monotone map and F is a set-valued map, with images not necessarily convex.

Introduction

Differential inclusions of the form $\dot{x} \in -Ax + F(t,x)$ where A is a maximal monotone operator and F is Lipschitzian in x, have been considered in [3]. In [2] the single-valued perturbation is replaced by a set-valued map, convex and upper semicontinuous and the existence of solutions is proved by a fixed point approach based on Kakutani's theorem.

In [1] inclusions of the form $\dot{x} \in F(t,x)$, i.e. with $A = 0$, have been treated, with no convexity assumptions on F, by a selection argument and by the use of Schauder's theorem. It is our purpose to present an existence theorem for differential inclusions

$$\dot{x} \in -Ax + F(t,x)$$

where A is a maximal monotone operator and F is continuous but not necessarily convex-valued.

We wish to point out that, although the approach is the same selection approach as in [1], the properties of solutions of a differential equation with a monotonic right hand side force us to prove the existence of a selection, continuous from L^1 into L^1. The procedure of [1] would not, in general, yield a selection enjoying this property.

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1. Notations and preliminary results

In the following $|\cdot|$, $\langle\cdot,\cdot\rangle$ and **d** will denote respectively the norm, the inner product and the Hausdorff metric induced by $|\cdot|$ on the space of non-empty compact subsets of R^n. For S a subset of R^n, $|S| \doteq \sup\{|s| : s \in S\}$.

Let I be a compact interval of R. By $L^1(I)$, $L^\infty(I)$, $C(I)$ we will denote respectively the sets of integrable, essentialy bounded and continuous functions from I into R^n; by $\|\cdot\|_{1,I}$ (whenever there is no ambiguity by $\|\cdot\|_1$) the norm in $L^1(I)$; by $\|\cdot\|_{\infty,I}$ (resp. $\|\cdot\|_\infty$) the norm in $L^\infty(I)$ and $C(I)$ and by B (resp. $\mathring{B}$) the closed (resp. open) unit ball in $L^\infty(I)$.

Let $a \in R$ and $\{T_i\}$ be an increasing unbounded sequence of positive numbers. Set $J_i = [a, a+T_i]$. By $L^1_{\mathrm{loc}}[a,\infty)$ we will denote the space of functions from $[a,\infty)$ into R^n locally integrable on $[a,\infty)$ provided with the topology induced by the family of seminorms $\|\cdot\|_{1,J_i}$.

Let A be a maximal monotone operator in R^n, i.e. a map from a subset $D(A)$ of R^n into the subsets of R^n such that

$$\langle x_1 - x_2, y_1 - y_2\rangle \geqq 0 \qquad \text{for } (x_i, y_i) \in A, \quad i = 1,2 \quad \text{and} \quad R(I+A) = R^n.$$

Let $a \in R$ and $x^0 \in \overline{D(A)}$. Consider the problems

$$(\mathrm{P}_1) \qquad \dot{x} \in -Ax + f(t), \qquad x(a) = x^0$$

and

$$(\mathrm{P}_2) \qquad \dot{x} \in -Ax + F(t,x), \qquad x(a) = x^0.$$

The following is known about (P_1).

THEOREM 1.1. *Let $f \in L^1_{\mathrm{loc}}[a,\infty)$. Then there exists a unique solution $u(f)$ to* (P_1)*, defined on $[a,\infty)$. The solution satisfies:*

$$\text{(i)} \qquad \|\dot{u}(f)\|_{1,I} \leqq C[(1 + T + \|f\|_{1,I})(1 + \|u(f)\|_{\infty,I}) + |u(\tau)|^2]$$

where C is a constant depending upon A, for every $I = [\tau, \tau + T]$ included in $[a,\infty)$.

Moreover for f and g in $L^1_{\mathrm{loc}}(a,\infty)$ and for every t

$$\text{(ii)} \qquad |u(f)(t) - u(g)(t)| \leqq \int_a^t |f(s) - g(s)|\, ds.$$

In particular, for $f \equiv 0$, (P_1) has a unique solution $u(0)(\cdot)$. Let i be the function from $L^1_{\mathrm{loc}}[a,\infty)$ into itself defined by $i(f) = u(f)$. The above Theorem in particular implies the following.

PROPOSITION 1.1. *The function i is continuous.*

Let F be a map from $[a,\infty)\times\overline{D(A)}$ into the subsets of R^n. By solution to (P_2) we mean a function u such that for some measurable selection $f(t)$ from $F(t,u(t))$, u is solution to (P_1).

We seek a solution to (P_2) defined on $[a,\infty)$. We assume that F satisfies the following.

ASSUMPTION (H). There exist two non-negative functions α, β, locally integrable on $[a,\infty)$, such that

$$|F(t,x)|\leqq \alpha(t)|x|+\beta(t). \qquad \blacksquare$$

Let I be a compact interval of R. Define the function $M_a : L^1(I)\to C(I)$ by

$$M_a(u)(t)=(2a)^{-1}\int_{-a}^{+a}u(t+s)ds.$$

PROPOSITION 1.2. *M_a is continuous.*

We shall need the following results about totally bounded subsets of $L^1(I)$ [4].

PROPOSITION 1.3. *A subset K of $L^1(I)$ is totally bounded iff it is bounded and*

$$\lim_{t\to 0}\int_I |u(t+s)-u(s)|ds=0$$

uniformly for u in K. In this case, for every $\varepsilon>0$ there exists $a>0$ such that

$$\sup\{\|M_a u-u\|_1 : u\in K\}<\varepsilon.$$

PROPOSITION 1.4. *K compact in $L^1(I)$ implies that*

$$\tilde{K}\doteq\Big(\bigcup_{a>0} M_a K\Big)\cup K$$

is compact in $L^1(I)$.

2. Main results

We wish to prove the following existence theorem for solutions to

$$(\mathrm{P}_2)\qquad \dot{x}\in -Ax+F(t,x),\qquad x(a)=x^0.$$

THEOREM. *Let A be a maximal monotone operator in R^n and F a continuous map from $[a,\infty)\times\overline{D(A)}$ into the compact subsets of R^n, satisfying assumption* (H). *Then there exists a solution to* (P_2) *defined on* $[a,\infty)$.

The proof of the above theorem rests upon a continuous selection argument (Theorem 2.1 below) and on Schauder's Fixed Point Theorem. We shall define a continuous function $g: L^1_{\mathrm{Loc}}[a,\infty) \to L^1_{\mathrm{Loc}}[a,\infty)$, a selection from F in the sense that for every u in a suitable subset of $L^1_{\mathrm{Loc}}[a,\infty)$ and a.e. t in $[a,\infty)$, $g(u)(t) \in F(t, u(t))$. Recalling the definition of the continuous map $i: L^1_{\mathrm{Loc}}[a,\infty) \to L^1_{\mathrm{Loc}}[a,\infty)$ given in section 1, as the function that associates to f the unique solution to (P_1), the problem will be reduced to that of finding a fixed point of the map

$$u \to i(g(u)).$$

For this purpose we shall define a compact and convex subset of $L^1_{\mathrm{Loc}}[a,\infty)$ mapped into itself by $i \circ g$. In the following Lemma we begin by a statement concerning the invariance of a given set. Recall that $u(0)$ is the solution to (P_1) corresponding to $f \equiv 0$.

LEMMA 2.1. *Let A be a maximal monotone operator in R^n and F a continuous map from $[a,\infty) \times D(A)$ into the compact subsets of R^n, satisfying assumption* (H). *Let w be a function from $[a,\infty)$ into R^n and $u:[a,\infty) \to R^n$ be a solution to* (P_1) *for $f(t)$ a measurable selection from $F(t, w(t))$. Then whenever w satisfies the inequality*

$$|w(t) - u(0)(t)| \leqq \int_a^t \gamma(s) e^{\int_s^t \alpha(l)dl} ds \tag{2.1}$$

where $\gamma(t) = \alpha(t)|u(0)(t)| + \beta(t)$, so does u.

PROOF. By assumption

$$|f(t)| \leqq \alpha(t)|w(t)| + \beta(t) \leqq \alpha(t)|w(t) - u(0)(t)| + \gamma(t)$$

hence, when w satisfies inequality (2.1),

$$\begin{aligned}
\int_a^t |f(s)|ds &\leqq \int_a^t \alpha(s) \int_a^s \gamma(l)\exp\left(\int_l^s \alpha(m)dm\right) dlds + \int_a^t \gamma(s)ds \\
&= \int_a^t \gamma(l) \int_l^t \alpha(s)\exp\left(\int_l^s \alpha(m)dm\right) dsdl + \int_a^t \gamma(s)ds \\
&= \int_a^t \gamma(l) \int_l^t \frac{d}{ds} \exp\left(\int_l^s \alpha(m)dm\right) dsdl + \int_a^t \gamma(s)ds \\
&= \int_a^t \gamma(l) \left[\exp\left(\int_l^t \alpha(m)dm\right) - 1\right] dl + \int_a^t \gamma(s)ds \\
&= \int_a^t \gamma(l)\exp\left(\int_l^t \alpha(m)dm\right) dl.
\end{aligned}$$

By Theorem 1.1 we know that

$$|u(t)-u(0)(t)| = |u(f)(t)-u(0)(t)| \leqq \int_0^t |f(s)|\,ds$$

concluding the proof. ■

LEMMA 2.2. *Let $I=[\tau,\tau+T]$ for some $T>0$. Let K be a subset of $L^1(I)$, bounded in $L^\infty(I)$. Let D be a subset of R^n containing the range of u for each u in K. Let F be a continuous function from $I\times D$ into the compact subsets of R^n satisfying assumption* (H). *Then for every $\varepsilon>0$ there exists δ such that for u and w in K, $\|u-w\|_1<\delta$ implies*

$$\int_I \mathbf{d}(F(t,u(t)),F(t,w(t)))dt<\varepsilon.$$

PROOF. Let P be such that for every u in K, $\|u\|_\infty \leqq P$. Let η be such that for $|x|\leqq P$ and $|y|\leqq P$, $|x-y|<\eta$ implies $\sup\{d(F(t,x),F(t,y)): t\in I\}<\varepsilon/3T$ and

$$\int_E [P\alpha(t)+\beta(t)]dt<\varepsilon/3$$

whenever $E\subset I$ and $m(E)\leqq\eta$.

For u and w in K, define $E=E(u,w)=\{t\in I: |u(t)-w(t)|>\eta\}$. Whenever $\|u-w\|_1<\eta^2 \doteq \delta$, $m(E)<\eta$. Hence

$$\begin{aligned}
&\int_I \mathbf{d}(F(t,u(t)),F(t,w(t)))dt \\
&= \int_E \mathbf{d}(F(t,u(t)),F(t,w(t)))dt + \int_{I\setminus E} \mathbf{d}(F(t,u(t)),F(t,w(t)))dt \\
&\leqq \int_E \{\alpha(t)(|u(t)|+|w(t)|)+2\beta(t)\}dt + \varepsilon/3Tm(I\setminus E) \\
&< 2\varepsilon/3+\varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

■

The following is the result on the existence of a continuous selection.

THEOREM 2.1. *Let K, D and F be as in Lemma 2.2. Assume moreover that K is compact and D convex. Then there exists a continuous function $h: K\to L^1(I)$ such that for each u in K, $h(u)(t)\in F(t,u(t))$ a.e. on I.*

PROOF. Actually we shall have to define h on the compact $\tilde{K} \supset K$. Remark that the convexity of D implies that $F(t, u(t))$ is defined for every u in $\tilde{K}$. Let us define a sequence of functions $g^i : \tilde{K} \to L^1(I)$ satisfying:

(a) $\int_I d(g^i(u)(t), F(t, u(t)))dt < 2^{-i}$ for every u in $\tilde{K}$;

(b) for every η, for every i, there exists ρ_i such that for every u in K, $\| u - w \|_1 < \rho_i$ implies $g^i(u)(t) = g^i(w)(t)$ at every t in $I \setminus E(u)$, where $E(u)$ is a finite union of subintervals of I such that $m(E(u)) < \eta$.

(c) $\int_I | g^{i+1}(u)(t) - g^i(u)(t) | dt < 2^{-(i-1)}$ for every u in $\tilde{K}$.

Since $K \subset PB$, then $\tilde{K} \subset PB$. Let Δ_0 be such that for $|x| \leqq P$ and $|y| \leqq P$, $|x - y| < \Delta_0$ implies $\sup\{\mathbf{d}(F(t, x), F(t, y)) : t \in I\} < 1/3T$, and δ_0 be such that for u and w in K, $\| u - w \|_1 < \delta_0$ implies

$$\int_I \mathbf{d}(F(t, u(t)), F(t, w(t)))dt < \tfrac{1}{3}.$$

Choose $a(0)$ such that $\sup\{\| M_{a(0)}u - u \|_1 : u \in \tilde{K}\} < \delta_0$, set for simplicity $M = M_{a(0)}$ and define, for each u in $\tilde{K}$:

$$0^0(u) = \{w \in \tilde{K} : \| Mu - Mw \|_\infty < \Delta_0\} = M^{-1}(Mu + \Delta_0 \mathring{B}).$$

By Proposition 1.2, $0^0(u)$ is open. Let $\{0^0(u_i^0)\}$ be a finite subcover of $\{0^0(u) : u \in K\}$. Let $\{I_i^0\}$ be the partition of I defined by setting, following [1], for each u in $\tilde{K}$

$$I_1^0(u) = [\tau, \tau + Tp_1(u)[, \qquad I_i^0(u) = \left[\tau + T\sum_{j=1}^{i-1} p_j(u), \tau + T\sum_{j=1}^{i} p_j(u)\right[$$

for $i \geqq 2$, where $\{p_i^0\}$ is the partition of unity subordinate to $\{0^0(u_i^0)\}$.

Choose $v_i^0(t)$, a measurable selection from $F(t, Mu_i^0)$, and define, for each u in $\tilde{K}$

$$g^0(u)(t) = \sum_i v_i^0(t)\chi_i^0(u)(t)$$

where $\chi_i^0(u)$ is the characteristic function of $I_i^0(u)$.

Fix u in $\tilde{K}$. Set for simplicity, $I_i = I_i^0(u)$. Then

$$\begin{aligned}
&\int_{I_i} d(g^0(u)(t), F(t, u(t)))dt \\
&\leqq \int_{I_i} d(v_i^0(t), F(t, Mu_i^0(t)))dt + \int_{I_i} \mathbf{d}(F(t, Mu_i^0(t)), F(t, Mu(t)))dt \\
&\quad + \int_{I_i} \mathbf{d}(F(t, Mu(t)), F(t, u(t)))dt.
\end{aligned}$$

The first integral is zero, the second is bounded by $m(I_i)/3T$, hence

(a) $\int_I d(g^0(u)(t), F(t, u(t)))dt < \frac{1}{3} + \int_I \mathbf{d}(F(t, Mu(t)), F(t, u(t)))dt < \frac{2}{3} < 1$,

(b) follows from the uniform continuity of $\{p_i^0\}$.

Assume we have defined g^i satisfying (a) and (b) above up to $i = n$. Define g^{n+1} satisfying (a), (b) and (c) as follows. Let η be such that

$$\int_E [P\alpha(t) + \beta(t)]dt < 2^{-(n+1)}/12,$$

whenever $E \subset I$ and $m(E) < \eta$. Let δ_{n+1} be such that for every u in $\tilde{K}$, $\|u - w\|_1 < \delta_{n+1}$ implies at once

$$\int_I \mathbf{d}(F(t, u(t)), F(t, w(t)))dt < 2^{-(n+1)}/3$$

and $g^n(u)(t) = g^n(w)(t)$ at every t in $I \setminus E(u)$, where $E(u)$ is a finite union of subintervals of I such that $m(E(u)) < \eta$. Choose $\Delta_{n+1} < \delta_{n+1}/T$ such that for $|x| \leqq P$ and $|y| \leqq P$, $|x - y| < \Delta_{n+1}$ implies

$$\sup\{\mathbf{d}(F(t, x), F(t, y)) : t \in I\} < 2^{-(n+1)}/3T.$$

Choose $a(n+1)$ so that:

$$\sup\{\|M_{a(n+1)}u - u\|_1 : u \in \tilde{K}\} < \delta_{n+1}.$$

Set for simplicity $M \doteq M_{a(n+1)}$ and define, for each u in $\tilde{K}$:

$$0^{n+1}(u) = \{w \in K : \|Mu - Mw\|_\infty < \Delta_{n+1}\} = M^{-1}(Mu + \Delta_{n+1}\mathring{B}).$$

Let $\{0^{n+1}(u_i^{n+1})\}$ be a finite subcover of this open cover of $\tilde{K}$. For each i choose $v_i^{n+1}(t)$, a measurable selection from $F(t, Mu_i^{n+1}(t))$, such that for almost every t in I

$$|g^n(Mu_i^{n+1})(t) - v_i^{n+1}(t)| = d(g^n(Mu_i^{n+1})(t), F(t, Mu_i^{n+1}(t))).$$

Let I_i^{n+1} be the partition of I defined, as before, by a partition of unity associated to $\{0^{n+1}(u_i^{n+1})\}$. Set $\chi_i^{n+1}(u)$ the characteristic function of the interval $I_i^{n+1}(u)$ and define

$$g^{n+1}(u)(t) = \sum_i v_i^{n+1}(t)\chi_i^{n+1}(u)(t).$$

Fix u in K. Set for simplicity $I_i = I_i^{n+1}(u)$.

(a) $$\int_{I_i} d(g^{n+1}(u)(t), F(t,u(t)))dt$$

$$< \int_{I_i} d(v_i^{n+1}(t), F(t, Mu_i^{n+1}(t)))dt + \int_{I_i} \mathbf{d}(F(t, Mu_i^{n+1}(t)), F(t, Mu(t)))dt$$

$$+ \int_{I_i} \mathbf{d}(F(t, Mu(t)), F(t, u(t)))dt.$$

The first integral is zero, the second, by definition of $0^{n+1}(u_i^{n+1})$, is bounded by $m(I_i)2^{-(n+1)}/3T$. Hence

$$\int_I d(g^{n+1}(u)(t), F(t,u(t)))dt < 2^{-(n+1)}/3 + \int_I \mathbf{d}(F(t, Mu(t)), F(t, u(t)))dt$$

$$< 2\frac{2^{-(n+1)}}{3}$$

$$< 2^{-(n+1)}.$$

(b) As before.

(c) $$\int_{I_i} |g^{n+1}(u)(t) - g^n(u)(t)|\, dt$$

$$= \int_{I_i} |v_i^{n+1}(t) - g^n(Mu_i^{n+1})(t) + g^n(Mu_i^{n+1})(t) - g^n(Mu)(t)$$

$$+ g^n(Mu)(t) - g^n(u)(t)|\, dt$$

$$\leqq \int_{I_i} d(g^n(Mu_i^{n+1})(t), F(t, Mu_i^{n+1}(t)))dt$$

$$+ \int_{I_i} |g^n(Mu_i^{n+1})(t) - g^n(Mu)(t)|\, dt$$

$$+ \int_{I_i} |g^n(Mu)(t) - g^n(u)(t)|\, dt.$$

Moreover:

$$\int_{I_i} d(g^n(Mu_i^{n+1})(t), F(t, Mu_i^{n+1}(t)))dt$$

$$\leqq \int_{I_i} |g^n(Mu_i^{n+1})(t) - g^n(Mu)(t)|\, dt + \int_{I_i} d(g^n(Mu)(t), F(t, Mu(t)))dt$$

$$+ \int_{I_i} \mathbf{d}(F(t, Mu(t)), F(t, Mu_i^{n+1}(t)))dt.$$

Hence

$$\int_I |g^{n+1}(u)(t)-g^n(u)(t)dt|$$

$$= \sum_i \int_{I_i} |g^{n+1}(u)(t)-g^n(u)(t)|\,dt$$

(2.2)

$$\leqq 2\sum_i \int_{I_i} |g^n(Mu_i^{n+1})(t)-g^n(Mu)(t)|\,dt + \int_I d(g^n(Mu)(t),F(t,Mu(t)))dt$$

$$+\sum_i \int_{I_i} \mathbf{d}(F(t,Mu(t)),F(t,Mu_i(t)))dt + \int_I |g^n(Mu)(t)-g^n(u)(t)|\,dt.$$

When $m(I_i)>0$, then $\|Mu_i^{n+1}-Mu\|_\infty<\Delta_{n+1}$, thus

$$\sum_i \int_{I_i} \mathbf{d}(F(t,Pu(t)),F(t,Mu_i^{n+1}(t)))dt < \sum_i \frac{2^{-(n+1)}}{3T}\cdot m(I_i) < \frac{2^{-(n+1)}}{3}.$$

The choice of Δ_{n+1} implies $\|Mu_i^{n+1}-Mu\|_1<\delta_{n+1}$, thus $g^n(Mu_i^{n+1})(t)=g^n(Mu)(t)$ at every t in $I\setminus E(Mu)$ and $m(E(Mu))<\eta$. As a consequence

$$2\sum_i \int_{I_i} |g^n(Mu_i^{n+1})(t)-g^n(Mu)(t)|\,dt < 2\int_{E(Mu)} 2(P\alpha(t)+\beta(t))dt$$

$$< \frac{2^{-(n+1)}}{3}.$$

In the same way we prove

$$\int_I |g^n(Mu)(t)-g^n(u)(t)|\,dt < \frac{2^{-(n+1)}}{6}.$$

Thus (2.2) becomes

$$\int_I |g^{n+1}(u)(t)-g^n(u)(t)|\,dt < 2\cdot 2^{-(n+1)}/3 + 2^{-(n+1)}/6$$

$$+\int_I d(g^n(Mu)(t),F(t,Mu(t)))dt.$$

Point (a) of the induction implies that the last term is bounded by 2^{-n}, hence

$$\int_I |g^{n+1}(u)(t)-g^n(u)(t)|\,dt < 2^{-n-1}+2^{-n} < 2^{-(n-1)}.$$

Note that from (b) it follows that every $g^i:\tilde{K}\to L^1(I)$ is continuous. Indeed fix $\varepsilon>0$ and choose η so that $\int_E [P\alpha(t)+\beta(t)]dt<\varepsilon/2$ whenever $E\subset I$ and $m(E)<\eta$. Set ρ_i as in (b). Then for every u in $\tilde{K}$, $\|u-w\|_1<\rho_i$ implies

$$\int_I |g^n(u)(t)-g^n(w)(t)|\,dt = \int_{E(u)} |g^n(u)(t)-g^n(w)(t)|\,dt$$

$$< 2\int_{E(u)} [P\alpha(t)+\beta(t)]dt$$

$$< \varepsilon.$$

The sequence of functions g^i is, by (c), a Cauchy sequence, thus it converges to a continuous function h. For every u in K, the sequence $g^i(u)$ converges in $L^1(I)$ to $h(u)$, hence there exists a subsequence which converges pointwise to $h(u)$ for almost every t in K. Since $d(g^i(u)(t), F(t,u(t)))$ converges to 0 and $F(t,u(t))$ is closed, $h(u)(t)\in F(t,u(t))$ for almost every t in I. ■

THEOREM 2.2. *Let A and F be as in Lemma* 2.1. *Then there exists a solution to* (P_2) *defined on* $[a,\infty)$.

PROOF. By the above results the proof consists in defining a compact convex set K and a suitable map transforming K into itself. Let K be the subset of $L^1_{\mathrm{loc}}[a,\infty)$ consisting of those absolutely continuous u satisfying:

(i) $u(a)=x^0$ and $u(t)\in\overline{D(A)}$,

(ii) $|u(t)-u(0)(t)|\leqq \int_a^t \gamma(s)\exp(\int_s^t \alpha(l)dl)ds$,

and

(iii) $\|\dot u\|_{1,I}\leqq C[(1+T+N(I))(1+M(I))+|r(\tau)^2|]$ for every $I=[\tau,\tau+T]$, $\tau\geqq a$, where we set

$$M(I)=\exp\left(\int_I \alpha(t)dt\right)\cdot\int_I \gamma(t)dt+\|u(0)(\cdot)\|_{\infty,I},$$

$$N(I)=M(I)\int_I \alpha(t)dt+\int_I \beta(t)dt,$$

and

$$r(\tau)=|u(0)(\tau)|+2\int_a^\tau \gamma(t)\exp\left(2\int_t^\tau \alpha(s)ds\right)dt.$$

K is nonempty since it contains $u(0)(\cdot)$; it is convex since so does $D(A)$; it is compact since, for each I, the set $K_I=\{u|_I : u\in K\}$ is, by Proposition 1.3, compact in $L^1(I)$. Moreover, by (ii), K_I is bounded in $L^\infty(I)$.

Set $I_1 \doteq J_1$, $I_i\doteq \overline{J_i\setminus J_{i-1}}$ for $i=2,3,\cdots$. By Theorem 2.1, we can define a family of continuous functions $h_i : K_{I_i}\to L^1(I_i)$ such that, for each u in K,

$$g_i(u|_{I_i})(t)\in F(t,u(t)) \qquad \text{a.e. on } I_i.$$

Thus the function $g : L^1_{loc}(a,\infty) \to L^1_{loc}(a,\infty)$ defined by

$$g(u)(t) = \begin{cases} h_1(u|_{I_1})(t) & t \in I_1, \\ h_i(u|_{I_i})(t) & t \in J_i \setminus J_{i-1}, \quad i = 2,3,\cdots \end{cases}$$

is continuous and satisfies $g(u(t)) \in F(t, u(t))$, a.e. on R.

Define s from K into $L^1_{loc}(a,\infty)$ by $s(u) = i(g(u))$. The continuity of s follows from Proposition 1.1.

Since for every u, $g(u)$ is a selection from $F(t, u(t))$, by Lemma 2.1, $s(u)$ satisfies (ii). Thus in particular it follows that $\|s(u)\|_{\infty,I} \leqq M(I)$ and $|s(u)(\tau)| \leqq r(\tau)$. By assumption (H), $\|g(u)\|_{1,I} \leqq N(I)$. Hence, by Theorem 1.1 (i), s satisfies (iii). So K is invariant under s, and, by Schauder's Theorem, it has a fixed point, a solution to (P_1) defined on $[a,\infty)$. ∎

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